

# Noise Kernel in Stochastic Gravity and Stress Energy Bi-Tensor of Quantum Fields in Curved Spacetimes

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The noise kernel is the vacuum expectation value of the (operator-valued) stress-energy bi-tensor which describes the fluctuations of a quantum field in curved spacetimes. It plays the role in stochastic semiclassical gravity based on the Einstein-Langevin equation similar to the expectation value of the stress-energy tensor in semiclassical gravity based on the semiclassical Einstein equation. According to the stochastic gravity program, this two point function (and by extension the higher order correlations in a hierarchy) of the stress energy tensor possesses precious statistical mechanical information of quantum fields in curved spacetime and, by the self-consistency required of Einstein's equation, provides a probe into the coherence properties of the gravity sector (as measured by the higher order correlation functions of gravitons) and the quantum and/or extended nature of spacetime. It reflects the medium energy (referring to Planck energy as high energy) or mesoscopic behavior of any viable theory of quantum gravity, including string theory. The stress energy bi-tensor could be the starting point for a new quantum field theory constructed on spacetimes with extended structures. In the coincidence limit we use the method of point-separation to derive a regularized noise-kernel for a scalar field in general curved spacetimes. It is useful for calculating quantum fluctuations of fields in modern theories of structure formation and for backreaction problems in the early universe and black holes. One collorary of our finding is that for a massless conformal field the trace of the noise kernel identically vanishes. We outline how the general framework and results derived here can be used for the calculation of noise kernels for specific cases of physical interest such as the Robertson-Walker and Schwarzschild spacetimes.

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## I. INTRODUCTION

The central focus of this work is the noise kernel, which is the vacuum expectation value of the stress-energy bi-tensor for a quantum field in curved spacetime [1–3]. It plays the role in stochastic semiclassical gravity [6,7] similar to the expectation value of the stress-energy tensor in semiclassical gravity [1]. We believe that this two point function (and the hierarchy of higher order correlation functions) of the stress energy tensor possesses precious statistical mechanical information about quantum fields in curved spacetime and reflects the low and medium energy (referring to Planck energy as high energy) behavior of any viable theory of quantum gravity, including string theory. The context for our investigation is stochastic gravity and the methodology is point-separation. Let us first examine the context and then the methodology.

### A. Stochastic Gravity

In semiclassical gravity the classical spacetime (with metric  $g_{ab}$ ) is driven by the expectation value  $\langle \rangle$  of the stress energy tensor  $T_{ab}$  of a quantum field with respect to some quantum state. One main task in the 70's was to obtain a regularized expression for this quantum source of the semiclassical Einstein equation (SCE) (e.g., [4,5] and earlier work referred therein). In stochastic semiclassical gravity of the 90's [6,7] the additional effect of fluctuations of the stress energy tensor [8–10] enters which induces metric fluctuations in the classical spacetime described by the Einstein-Langevin equation (ELE) [11,12]. This stochastic term measures the fluctuations of quantum sources (e.g., arising from the difference of particles created in neighboring histories [11]) and is intrinsically linked to the dissipation in the dynamics of spacetime by a fluctuation-dissipation relation [13,14], which embodies the full backreaction effects of quantum fields on classical spacetime.

The stochastic semiclassical Einstein equation, or Einstein-Langevin equation, takes on the form

$$G_{ab}[g] + \Lambda g_{ab} = 8\pi G(T_{ab}^c + T_{ab}^{qs})$$

$$T_{ab}^{qs} \equiv \langle T_{ab} \rangle_q + T_{ab}^s \quad (1.1)$$

where  $G_{ab}$  is the Einstein tensor associated with  $g_{ab}$  and  $\Lambda, G$  are the cosmological and Newton constants respectively. Here we use the superscripts c, s, q to denote classical, stochastic and quantum respectively. The new term  $T_{ab}^s = 2\tau_{ab}$  which is of classical stochastic nature measures the fluctuations of the energy momentum tensor of the quantum field. To see what  $\tau_{ab}$  is, first define

$$\hat{t}_{ab}(x) \equiv \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle \hat{I} \quad (1.2)$$

which is a tensor operator measuring the deviations from the mean of the stress energy tensor. We are interested in the correlation of these operators at different spacetime points. Here we focus on the stress energy (operator-valued) bi-tensor  $\hat{t}_{ab}(x)\hat{t}_{c'd'}(y)$  defined at nearby points  $(x, y)$ . metric.) A bi-tensor is a geometric object that has support at two separate spacetime points. In particular, it is a rank two tensor in the tangent space at  $x$  (with unprimed indices) and in the tangent space at  $y$  (with primed indices).

The noise kernel  $N_{abc'd'}$  bitensor is defined as

$$4N_{abc'd'}(x, y) \equiv \frac{1}{2} \langle \{ \hat{t}_{ab}(x), \hat{t}_{c'd'}(y) \} \rangle \quad (1.3)$$

where  $\{ \}$  means taking the symmetric product. In the influence functional (IF) [15] or closed-time-path (CTP) effective action approach [16] the noise kernel appears in the real part of the influence action [11,17].<sup>1</sup> The noise

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<sup>1</sup>The derivation of noise kernel from IF or CTP coarse-grained effective action is well-known from earlier work [11,13,14,7]. Generally speaking the effective action approach excels in the treatment of backreaction effects of quantum fields on background spacetimes since it has self-consistency built in. However, most calculations of the influence actions derived so far require some form of perturbative expansion, because it is difficult to obtain the influence functional in closed form, (exceptions are only for spacetimes with high symmetry, and for special classes of coupling between the field and spacetime). We adopt the covariant point separation method because we can obtain an expression for the noise kernel in general curved spacetime with explicit dependence on the world function between two points and its covariant derivatives. This will be useful for later exploration of extended feature of spacetime.

kernel defines a real classical Gaussian stochastic symmetric tensor field  $\tau_{ab}$  which is characterized to lowest order by the following relations,

$$\langle \tau_{ab}(x) \rangle_\tau = 0, \quad \langle \tau_{ab}(x) \tau_{c'd'}(y) \rangle_\tau = N_{abc'd'}(x, y), \quad (1.4)$$

where  $\langle \rangle_\tau$  means taking a statistical average with respect to the noise distribution  $\tau$  (for simplicity we don't consider higher order correlations). Since  $\hat{T}_{ab}$  is self-adjoint, one can see that  $N_{abc'd'}$  is symmetric, real, positive and semi-definite. Furthermore, as a consequence of (1.3) and the conservation law  $\nabla^a \hat{T}_{ab} = 0$ , this stochastic tensor  $\tau_{ab}$  is divergenceless in the sense that  $\nabla^a \tau_{ab} = 0$  is a deterministic zero field. Also  $g^{ab} \tau_{ab}(x) = 0$ , signifying that there is no stochastic correction to the trace anomaly (if  $T_{ab}$  is traceless). (See [7]). Here all covariant derivatives are taken with respect to the background metric  $g_{ab}$  which is a solution of the semiclassical equations. Taking the statistical average of (1.1) with respect to the noise distribution  $\tau$ , as a consequence of the noise correlation relation (1.4),

$$\langle T_{ab}^{qs} \rangle_\tau = \langle T_{ab} \rangle_q \quad (1.5)$$

we recover the semiclassical Einstein equation which is (1.1) without the  $T_{ab}^s$  term. It is in this sense that we view semiclassical gravity as a mean field theory.

## B. Relation to Semiclassical and Quantum Gravity

Stochastic semiclassical gravity thus ingrains a relation between noise in quantum fields and metric fluctuations. While the semiclassical regime describes the effect of a quantum matter field only through its mean value (e.g., vacuum expectation value), the stochastic regime includes the effect of fluctuations and correlations. We believe precious new information resides in the two-point functions of the stress energy tensor which may reflect the finer structure of spacetime at a scale when information provided by its mean value as source (semiclassical gravity) is no longer adequate. To appreciate this, it is perhaps instructive to examine the distinction among these three theories: stochastic gravity in relation to semiclassical and quantum gravity [7,6]. The following observation will also bring out two other related concepts of correlation (in the quantum field) and coherence (in quantum gravity).

### 1. Classical, Stochastic and Quantum

For concreteness we consider the example of gravitational perturbations  $h_{ab}$  in a background spacetime with metric  $g_{ab}$  driven by the expectation value of the energy momentum tensor of a scalar field  $\Phi$ , as well as its fluctuations  $\hat{t}_{ab}(x)$  [13,14]. Let us compare the stochastic with the semiclassical and quantum equations of motion for the metric perturbation (weak but deterministic) field  $h$ . (This schematic representation was made by E. Verdaguer in [7]). The semiclassical equation is given by

$$\square h = 16\pi G \langle \hat{T} \rangle \quad (1.6)$$

where  $\langle \rangle$  denotes taking the quantum average (e.g., the vacuum expectation value) of the operator enclosed. Its solution can be written in the form

$$h = \int C \langle \hat{T} \rangle, \quad h_1 h_2 = \int \int C_1 C_2 \langle \hat{T} \rangle \langle \hat{T} \rangle. \quad (1.7)$$

The quantum (Heisenberg) equation

$$\square \hat{h} = 16\pi G \hat{T} \quad (1.8)$$

has solutions

$$\hat{h} = \int C \hat{T}, \quad \langle \hat{h}_1 \hat{h}_2 \rangle = \int \int C_1 C_2 \langle \hat{T} \hat{T} \rangle_{\hat{h}, \hat{\phi}} \quad (1.9)$$

where the average is taken with respect to the quantum fluctuations of both the gravitational ( $\hat{g}$ ) and the matter ( $\hat{\phi}$ ) fields. Now for the stochastic equation, we have

$$\square h = 16\pi G (\langle \hat{T} \rangle + \tau) \quad (1.10)$$

with solutions <sup>2</sup>

$$h = \int C \langle \hat{T} \rangle + \int C \tau, \quad h_1 h_2 = \int \int C_1 C_2 [\langle \hat{T} \rangle \langle \hat{T} \rangle + (\langle \hat{T} \rangle \tau + \tau \langle \hat{T} \rangle) + \tau \tau] \quad (1.11)$$

Now take the noise average  $\langle \rangle_\tau$ . Recall that the noise is defined in terms of the stochastic sources  $\tau$  as

$$\langle \tau \rangle_\tau = 0, \quad \langle \tau_1 \tau_2 \rangle_\tau \equiv \langle \hat{T}_1 \hat{T}_2 \rangle - \langle \hat{T}_1 \rangle \langle \hat{T}_2 \rangle \quad (1.12)$$

we get

$$\langle h_1 h_2 \rangle_\tau = \int \int C_1 C_2 \langle \hat{T} \hat{T} \rangle_{\hat{\phi}} \quad (1.13)$$

Note that the correlation of the energy momentum tensor appears just like in the quantum case, but the average here is over noise from quantum fluctuations of the matter field alone.

## 2. Fluctuations, Correlations and Coherence

Comparing the equations above depicting the semiclassical, stochastic and quantum regimes, <sup>3</sup> we see first that in the semiclassical case, the classical metric correlations is given by the product of the vacuum expectation value of the energy momentum tensor whereas in the quantum case it is given by the quantum average of the correlation of metric (operators) with respect to the fluctuations in both the matter and the gravitational fields. In the stochastic case the form is closer to the quantum case except that the quantum average is replaced by the noise average, and the average of the energy momentum tensor is taken with respect only to the matter field. The important improvement over semiclassical gravity is that it now carries information on the correlation of the energy momentum tensor of the fields and its induced metric fluctuations. Thus stochastic gravity contains information about the correlation of fields (and the related phase information) which is absent in semiclassical gravity. Here we have invoked the relation between *fluctuations* and *correlations*, a variant form of the fluctuation-dissipation relation. This feature moves stochastic gravity closer than semiclassical gravity to quantum gravity in that the correlation in quantum field and geometry fully present in quantum gravity is partially retained in stochastic gravity, and the background geometry has a way to sense the correlation of the quantum fields through the noise term in the Einstein-Langevin equation, which shows up as metric fluctuations.

By now we can see that ‘noise’ as used in this more precise language and broader context is not something one can arbitrarily assign or relegate, as is often done in ordinary discussions, but it has taken on a deeper meaning in that it embodies the contributions of the higher correlation functions in the quantum field. It holds the key to probing the quantum nature of spacetime in this vein. We begin our studies here with the lowest order term, i.e., the 2 point function of the energy momentum tensor which contains the 4th order correlation of the quantum field (or gravitons when they are considered as matter source).<sup>4</sup> Progress is made now on how to characterize the higher order correlation functions of an interacting field systematically from the Schwinger-Dyson equations in terms of ‘correlation noise’ [19,20], after the BBGKY hierarchy. This may prove useful for a correlation dynamics /stochastic semiclassical approach to quantum gravity [6].

Thus noise carries information about the correlations of the quantum field. One can further link *correlation* in quantum fields to *coherence* in quantum gravity. This stems from the self-consistency required in the backreaction

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<sup>2</sup>In this schematic form we have not displayed the homogeneous solution carrying the information of the (maybe random) initial condition. This solution will exist in general, and may even be dominant if dissipation is weak. When both the uncertainty in initial conditions and the stochastic noise are taken into account, the Einstein - Langevin formalism reproduces the exact graviton two point function, in the linearized approximation. Of course, it fails to reproduce the expectation value of observables which could not be written in terms of graviton occupation numbers, and in this sense it falls short of full quantum gravity. This comment was made by E. Calzetta to the author of [6].

<sup>3</sup>The observations in this section were first made in [6]

<sup>4</sup>Although the Feynman- Vernon way can only accomodate Gaussian noise of the matter fields and takes a simple form for linear coupling to the background spacetime, the notion of noise can be made more general and precise. For an example of more complex noise associated with more involved backreactions arising from strong or nonlocal couplings, see Johnson and Hu [18]

equations for the matter and spacetime sectors. The Einstein-Langevin equation is only a partial (low energy) representation of the complete theory of quantum gravity and fields. There, the coherence in the geometry is related to the coherence in the matter field, as the complete quantum description should be given by a coherent wave function of the combined matter and gravity sectors. Semiclassical gravity forsakes all the coherence in the quantum gravity sector. Stochastic gravity captures only partial coherence in the quantum gravity sector via the correlations in the quantum fields. Since the degree of coherence can be measured in terms of correlations, our strategy for the semiclassical stochastic gravity program is to unravel the higher correlations of the matter field, starting with the variance of the stress energy tensor and through its linkage with gravity, retrieve whatever quantum attributes (partial coherence) of gravity left over from the high energy behavior above the Planck scale. Thus in this approach, focussing on the noise kernel and the stress energy tensor two point function is our first step beyond mean field (semiclassical gravity) theory towards probing the full theory of quantum gravity.

### C. Point Separation and Noise Kernel

So far we have explained the physical motivation for investigating the noise kernel and stress energy bi-tensor. We now turn to the methodology.

In the light of the above discussions, the point separation scheme introduced in the 60's by DeWitt [21] will be well suited for our purpose here. It was brought to more popular use in the 70's in the context of quantum field theory in curved spacetimes [22–24] as a means for obtaining a finite quantum stress tensor. Since the stress tensor is built from the product of a pair of field operators evaluated at a single point, it is not well-defined. In this scheme, one introduces an artificial separation of the single point  $x$  to a pair of closely separated points  $x$  and  $x'$ . The problematic terms involving field products such as  $\hat{\phi}(x)^2$  becomes  $\hat{\phi}(x)\hat{\phi}(x')$ , whose expectation value is well defined. One then brings the two points back (taking the coincidence limit) to identify the divergences present, which will then be removed (regularization) or moved (by renormalizing the coupling constants), thereby obtaining a well-defined, finite stress tensor at a single point. In this context point separation was used as a technique (many practitioners may still view it as a trick, even a clumsy one) for the purpose of identifying the ultraviolet divergences.

By contrast, in our program, point separated expression of stress energy bi-tensor have fundamental physical meaning as it contains information on the fluctuations and correlation of quantum fields, and by consistency with the gravity sector, can provide a probe into the coherent properties of quantum spacetimes. Taking this view, we may also gain a new perspective on ordinary quantum field theory defined on single points: The coincidence limit depicts the low energy limit of the full quantum theory of matter and spacetimes. Ordinary (pointwise) quantum field theory, classical general relativity and semiclassical gravity are the lowest levels of approximations and should be viewed not as fundamental, but only as effective theories. As such, even the way how the conventional point-defined field theory emerges from the full theory when the two points (e.g.,  $x$  and  $y$  in the noise kernel) are brought together is interesting. For example, one can ask if there will also be a quantum to classical transition in spacetime accompanying the coincident limit? Certain aspects like decoherence has been investigated before (see, e.g., [25]), but here the non-local structure of spacetime and their impact on quantum field theory become the central issue. (This may also be a relevant issue in noncommutative geometry). The point-wise limit of field theory of course has ultraviolet divergences and requires regularization. A new viewpoint towards regularization evolved from this perspective of treating conventional pointwise field theory as an effective theory in the coincident limit of the point-separated theory of extended spacetime. This is discussed in our recent paper on fluctuations of the vacuum energy density and the validity of semiclassical gravity [10].

The paper is organized as follows: In Section II, we review the method of point separation. In Section III, we discuss the procedures for dealing with the quantum stress tensor bi-operator at two separated points and the noise kernel. We derive a general expression for the noise kernel in terms of the quantum field's Green function and its covariant derivatives up to the fourth order. (The stress tensor involves up to two covariant derivatives.) This result holds for  $x \neq y$  without recourse to renormalization of the Green function, showing that  $N_{abc'd'}(x, y)$  is always finite for  $x \neq y$  (and off the light cone for massless theories). Using this result, we show there is no stochastic correction to the trace anomaly, in agreement with results arrived at in [11,14]. In Section IV we briefly review how “modified” point separation [26–28] is used to get a regularized Green function. This amounts to subtracting a locally determined “Hadamard ansatz” from the Green function. With this we show how to compute the fluctuations of the stress tensor, culminating in a general expression of the noise kernel and its coincident form for an arbitrary curved spacetime. Paper II in this series will apply these formulas to ultrastatic metrics including the Einstein universe [9], hot flat space and optical Schwarzschild spacetimes. Paper III treats noise kernels in the Robertson-Walker universe and Schwarzschild black holes, from which structure formation from quantum fluctuations [29] and backreaction of Hawking radiation on the black hole spacetime [30] can be studied. Finally, we intend in Paper IV to use the symbolic routine to derive

or check on analytic expressions for the stress-energy bi-tensor in de Sitter and anti-de Sitter spacetimes. (Martin, Roura and Verdaguer [31] have obtained analytic expressions for Minkowski and restricted cases of de Sitter spaces.) The former is necessary for scrutinizing primordial fluctuations in the cosmic background radiation while the latter is related to black hole phase transition and AdS/CFT issues in string theory.

## II. POINT SEPARATION

We start with a short overview of point separation in this section. This paper will present the general schema and the results for the coincident limit of the noise kernel. The details, along with how they are carried out using symbolic computation, will be described in a separate paper by one of us [32]. Here we first review the construction of the stress tensor and then derive the symmetric stress tensor two point function, the noise kernel, in terms of the Wightman Green function. Our result is completely general for the case of a free scalar field (the separation is maintained throughout) and thus can be used with or without regularizing the Green function. Of course, without use of a regularized Green function, the  $y \rightarrow x$  limit is divergent. We also give the noise kernel for a massless conformally coupled free scalar field on a four dimensional manifold. We then compute the trace at both points and find this double trace vanishes identically for the massless conformal case; implying that there is no noise associated with the trace anomaly. This result holds separate from the issue of regularization.

Having the point-separated expression for the noise kernel and using the regularized Green function, as is done for the stress tensor, we obtain an expression for which the  $y \rightarrow x$  limit is meaningful. We use the so-call “modified” point separation prescription [26–28]. In this procedure, the naive Green function is rendered finite by assuming the divergences present for  $y \rightarrow x$  are state independent and can be removed by subtraction of a Hadamard form. We review this prescription and use it to obtain a definition of the noise kernel for which the  $y \rightarrow x$  limit is meaningful.<sup>5</sup>

### A. $n$ -tensors and end-point expansions

An object like the Green function  $G(x, y)$  is an example of a *bi-scalar*: it transforms as scalar at both points  $x$  and  $y$ . We can also define a *bi-tensor*  $T_{a_1 \dots a_n b'_1 \dots b'_m}(x, y)$ : upon a coordinate transformation, this transforms as a rank  $n$  tensor at  $x$  and a rank  $m$  tensor at  $y$ . We will extend this up to a *quad-tensor*  $T_{a_1 \dots a_{n_1} b'_1 \dots b'_{n_2} c''_1 \dots c''_{n_3} d'''_1 \dots d'''_{n_4}}$  which has support at four points  $x, y, x', y'$ , transforming as rank  $n_1, n_2, n_3, n_4$  tensors at each of the four points. This also sets the notation we will use: unprimed indices referring to the tangent space constructed above  $x$ , single primed indices to  $y$ , double primed to  $x'$  and triple primed to  $y'$ . For each point, there is the covariant derivative  $\nabla_a$  at that point. Covariant derivatives at different points commute and the covariant derivative at, say, point  $x'$ , does not act on a bi-tensor defined at, say,  $x$  and  $y$ :

$$T_{ab';c;d'} = T_{ab';d';c} \quad \text{and} \quad T_{ab';c''} = 0. \quad (2.1)$$

To simplify notation, henceforth we will eliminate the semicolons after the first one for multiple covariant derivatives at multiple points.

Having objects defined at different points, the coincident limit is defined as evaluation “on the diagonal”, in the sense of the spacetime support of the function or tensor, and the usual shorthand  $[G(x, y)] \equiv G(x, x)$  is used. This extends to  $n$ -tensors as

$$\left[ T_{a_1 \dots a_{n_1} b'_1 \dots b'_{n_2} c''_1 \dots c''_{n_3} d'''_1 \dots d'''_{n_4}} \right] = T_{a_1 \dots a_{n_1} b_1 \dots b_{n_2} c_1 \dots c_{n_3} d_1 \dots d_{n_4}}, \quad (2.2)$$

*i.e.*, this becomes a rank  $(n_1 + n_2 + n_3 + n_4)$  tensor at  $x$ . The multi-variable chain rule relates covariant derivatives acting at different points, when we are interested in the coincident limit:

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<sup>5</sup>We know from work in the 70’s in quantum field in curved spacetime [1–3] that there are several regularization methods developed for the removal of ultraviolet divergences in the stress energy tensor. Their mutual relations are known, and discrepancies explained. This formal structure of regularization schemes for quantum fields in curved spacetime should remain intact as we apply them to the regularization of the noise kernel in general curved spacetimes. Specific considerations will of course enter for each method. But for the methods we have employed so far, such as zeta-function, point separation, smeared-field [9,10] applied to simple cases (Casimir, Einstein, thermal fields) there is no new inconsistency or discrepancy.

$$[T_{a_1 \dots a_m b'_1 \dots b'_n}]_{;c} = [T_{a_1 \dots a_m b'_1 \dots b'_n}]_{;c} + [T_{a_1 \dots a_m b'_1 \dots b'_n}]_{;c'} . \quad (2.3)$$

This result is referred to as *Synge's theorem* in this context. (We follow Fulling's [2] discussion.)

The bi-tensor of *parallel transport*  $g_a^{b'}$  is defined such that when it acts on a vector  $v_{b'}$  at  $y$ , it parallel transports the vector along the geodesics connecting  $x$  and  $y$ . This allows us to add vectors and tensors defined at different points. We cannot directly add a vector  $v_a$  at  $x$  and vector  $w_{a'}$  at  $y$ . But by using  $g_a^{b'}$ , we can construct the sum  $v^a + g_a^{b'} w_{b'}$ . We will also need the obvious property  $[g_a^{b'}] = g_a^{b'}$ .

The main bi-scalar we need for this work is the *world function*  $\sigma(x, y)$ . This is defined as a half of the square of the geodesic distance between the points  $x$  and  $y$ . It satisfies the equation

$$\sigma = \frac{1}{2} \sigma^{;p} \sigma_{;p} \quad (2.4)$$

Often in the literature, a covariant derivative is implied when the world function appears with indices:  $\sigma^a \equiv \sigma^{;a}$ , *i.e.* taking the covariant derivative at  $x$ , while  $\sigma^{a'}$  means the covariant derivative at  $y$ . This is done since the vector  $-\sigma^a$  is the tangent vector to the geodesic with length equal the distance between  $x$  and  $y$ . As  $\sigma^a$  records information about distance and direction for the two points this makes it ideal for constructing a series expansion of a bi-scalar. The *end point* expansion of a bi-scalar  $S(x, y)$  is of the form

$$S(x, y) = A^{(0)} + \sigma^p A_p^{(1)} + \sigma^p \sigma^q A_{pq}^{(2)} + \sigma^p \sigma^q \sigma^r A_{pqr}^{(3)} + \sigma^p \sigma^q \sigma^r \sigma^s A_{pqrs}^{(4)} + \dots \quad (2.5)$$

where, following our convention, the expansion tensors  $A_{a_1 \dots a_n}^{(n)}$  with unprimed indices have support at  $x$  and hence the name end point expansion. Only the symmetric part of these tensors contribute to the expansion. For the purposes of multiplying series expansions it is convenient to separate the distance dependence from the direction dependence. This is done by introducing the unit vector  $p^a = \sigma^a / \sqrt{2\sigma}$ . Then the series expansion can be written

$$S(x, y) = A^{(0)} + \sigma^{\frac{1}{2}} A^{(1)} + \sigma A^{(2)} + \sigma^{\frac{3}{2}} A^{(3)} + \sigma^2 A^{(4)} + \dots \quad (2.6)$$

The expansion scalars are related to the expansion tensors via

$$A^{(n)} = 2^{n/2} A_{p_1 \dots p_n}^{(n)} p^{p_1} \dots p^{p_n}.$$

The last object we need is the *Van Vleck-Morette* determinant  $D(x, y)$ , defined as  $D(x, y) \equiv -\det(-\sigma_{;ab'})$ . The related bi-scalar

$$\Delta^{1/2} = \left( \frac{D(x, y)}{\sqrt{g(x)g(y)}} \right)^{\frac{1}{2}} \quad (2.7)$$

satisfies the equation

$$\Delta^{1/2} (4 - \sigma_{;p}{}^p) - 2\Delta^{1/2}{}_{;p} \sigma^{;p} = 0 \quad (2.8)$$

with the boundary condition  $[\Delta^{1/2}] = 1$ .

Further details on these objects and discussions of the definitions and properties are contained in [23] and [32]. There it is shown how the defining equations for  $\sigma$  and  $\Delta^{1/2}$  are used to determine the coincident limit expression for the various covariant derivatives of the world function ( $[\sigma_{;a}]$ ,  $[\sigma_{;ab}]$ , *etc.*) and how the defining differential equation for  $\Delta^{1/2}$  can be used to determine the series expansion of  $\Delta^{1/2}$ . We show how the expansion tensors  $A_{a_1 \dots a_n}^{(n)}$  are determined in terms of the coincident limits of covariant derivatives of the bi-scalar  $S(x, y)$ . Ref. [32] details how point separation can be implemented on the computer to provide easy access to a wider range of applications involving higher derivatives of the curvature tensors. We will say a few words about this to end this section.

## B. Symbolic Computation

Since the noise kernel involves up to four covariant derivatives, we expand the Green function out to fourth order in the distance between the points in an end-point expansion. One of the main advantages of this approach is that it readily lends itself to implementation in a symbolic computing environment [34,33]. So the task becomes one of developing the necessary series expansions of the various geometric objects, including the conformal transformation properties. From the symbolic computational viewpoint, the key to this procedure is to recognize that all the objects

for which a series expansion are needed are defined by first or second order covariant differential equations. This allows us to develop recursive algorithms for the symbolic manipulations. The lowest order terms in the series expansions are known [23,35]. For example, Christensen's work on the stress tensor goes to second order in  $\sigma^a$  (ending up creating results spanning a page and a half in his manuscript). The noise kernel results get much longer. (Some expansion coefficients end up over 400 terms in length.)<sup>6</sup>

We have developed a systematic set of computer code, using MathTensor [33], running in the Mathematica [34] environment on a workstation, to carry out much of the work contained in here and the following papers in this series. We use it to compute the fluctuations of the quantum stress tensor for scalar fields from a given form of the metric and an analytic expression for the Green function. We can also deal with cases where the Green function is only known in an analytic form in a conformally related spacetime. No recourse is made to numerical methods; the results are exact up to the accuracy of the given analytic form of the Green function *before* regularization. The code itself carries out the regularization via the “modified” point separation prescription.

### III. STRESS ENERGY BI-TENSOR OPERATOR AND NOISE KERNEL

Even though we believe that the point-separated results are more basic in the sense that it reflects a deeper structure of the quantum theory of spacetime, we will nevertheless start with quantities defined at one point because they are more familiar in conventional quantum field theory. We will use point separation to introduce the biquantities. The key issue here is thus the distinction between point-defined (*pt*) and point-separated (*bi*) quantities.

For a free classical scalar field with the action

$$S[\phi] = -\frac{1}{2} \int (m^2 \phi^2 + \phi^2 R \xi + \phi_{;p} \phi^{;p}) \sqrt{g} d^4x. \quad (3.1)$$

the classical stress energy tensor is

$$\begin{aligned} T_{ab}(x) &\equiv \frac{2}{\sqrt{g(x)}} \frac{\delta S[\phi]}{\delta g^{ab}(x)} \\ &= (1 - 2\xi) \phi_{;a} \phi_{;b} + \left(2\xi - \frac{1}{2}\right) \phi_{;p} \phi^{;p} g_{ab} + 2\xi \phi (\phi_{;p}^p - \phi_{;ab} g_{ab}) \\ &\quad + \phi^2 \xi \left(R_{ab} - \frac{1}{2} R g_{ab}\right) - \frac{1}{2} m^2 \phi^2 g_{ab} \end{aligned} \quad (3.2)$$

When we make the transition to quantum field theory, we promote the field  $\phi(x)$  to a field operator  $\hat{\phi}(x)$ . The fundamental problem of defining a quantum operator for the stress tensor is immediately visible: the field operator appears quadratically. Since  $\hat{\phi}(x)$  is an operator-valued distribution, products at a single point are not well-defined. But if the product is point separated ( $\hat{\phi}^2(x) \rightarrow \hat{\phi}(x)\hat{\phi}(x')$ ), they are finite and well-defined.

Let us first seek a point-separated extension of these classical quantities and then consider the quantum field operators. Point separation is symmetrically extended to products of covariant derivatives of the field according to

$$(\phi_{;a})(\phi_{;b}) \rightarrow \frac{1}{2} \left( g_a^{p'} \nabla_{p'} \nabla_b + g_b^{p'} \nabla_a \nabla_{p'} \right) \phi(x) \phi(x'), \quad (3.3)$$

$$\phi (\phi_{;ab}) \rightarrow \frac{1}{2} \left( \nabla_a \nabla_b + g_a^{p'} g_b^{q'} \nabla_{p'} \nabla_{q'} \right) \phi(x) \phi(x'). \quad (3.4)$$

The bi-vector of parallel displacement  $g_a^{a'}(x, x')$  is included so that we may have objects that are rank 2 tensors at  $x$  and scalars at  $x'$ .

To carry out point separation on (3.2), we first define the differential operator

$$\mathcal{T}_{ab} = \frac{1}{2} (1 - 2\xi) \left( g_a^{a'} \nabla_{a'} \nabla_b + g_b^{b'} \nabla_a \nabla_{b'} \right) + \left( 2\xi - \frac{1}{2} \right) g_{ab} g^{cd'} \nabla_c \nabla_{d'}$$

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<sup>6</sup>The final expressions reaching such a length defy direct inspection check. However, since the algorithms are recursive, agreement at the lower orders extends to higher orders for the correctness of the complete result. For insurance, some of the larger expressions are derived by two independent methods and verified to be the same.



$$\begin{aligned}
& -\xi \left( \nabla_a \nabla_b + g_a{}^{a'} g_b{}^{b'} \nabla_{a'} \nabla_{b'} \right) + \xi g_{ab} \left( \nabla_c \nabla^c + \nabla_{c'} \nabla^{c'} \right) \\
& + \xi \left( R_{ab} - \frac{1}{2} g_{ab} R \right) - \frac{1}{2} m^2 g_{ab}
\end{aligned} \tag{3.5}$$

from which we obtain the classical stress tensor as

$$T_{ab}(x) = \lim_{x' \rightarrow x} \mathcal{T}_{ab} \phi(x) \phi(x'). \tag{3.6}$$

That the classical tensor field no longer appears as a product of scalar fields at a single point allows a smooth transition to the quantum tensor field. From the viewpoint of the stress tensor, the separation of points is an artificial construct so when promoting the classical field to a quantum one, neither point should be favored. The product of field configurations is taken to be the symmetrized operator product, denoted by curly brackets:

$$\phi(x) \phi(y) \rightarrow \frac{1}{2} \left\{ \hat{\phi}(x), \hat{\phi}(y) \right\} = \frac{1}{2} \left( \hat{\phi}(x) \hat{\phi}(y) + \hat{\phi}(y) \hat{\phi}(x) \right) \tag{3.7}$$

With this, the point separated stress energy tensor operator is defined as

$$\hat{T}_{ab}(x, x') \equiv \frac{1}{2} \mathcal{T}_{ab} \left\{ \hat{\phi}(x), \hat{\phi}(x') \right\}. \tag{3.8}$$

While the classical stress tensor was defined at the coincidence limit  $x' \rightarrow x$ , we cannot attach any physical meaning to the quantum stress tensor at one point until the issue of regularization is dealt with, which will happen in the next section. For now, we will maintain point separation so as to have a mathematically meaningful operator.

The expectation value of the point-separated stress tensor can now be taken. This amounts to replacing the field operators by their expectation value, which is given by the Hadamard (or Schwinger) function

$$G^{(1)}(x, x') = \left\langle \left\{ \hat{\phi}(x), \hat{\phi}(x') \right\} \right\rangle. \tag{3.9}$$

and the point-separated stress tensor is defined as

$$\langle \hat{T}_{ab}(x, x') \rangle = \frac{1}{2} \mathcal{T}_{ab} G^{(1)}(x, x') \tag{3.10}$$

where, since  $\mathcal{T}_{ab}$  is a differential operator, it can be taken “outside” the expectation value. The expectation value of the point-separated quantum stress tensor for a free, massless ( $m = 0$ ) conformally coupled ( $\xi = 1/6$ ) scalar field on a four dimension spacetime with scalar curvature  $R$  is

$$\begin{aligned}
\langle \hat{T}_{ab}(x, x') \rangle = & \frac{1}{6} \left( g^{p'}{}_b G^{(1)}{}_{;p'a} + g^{p'}{}_a G^{(1)}{}_{;p'b} \right) - \frac{1}{12} g^{p'}{}_q G^{(1)}{}_{;p'q} g_{ab} \\
& - \frac{1}{12} \left( g^{p'}{}_a g^{q'}{}_b G^{(1)}{}_{;p'q'} + G^{(1)}{}_{;ab} \right) + \frac{1}{12} \left( \left( G^{(1)}{}_{;p'p'} + G^{(1)}{}_{;p'p} \right) g_{ab} \right) \\
& + \frac{1}{12} G^{(1)} \left( R_{ab} - \frac{1}{2} R g_{ab} \right)
\end{aligned} \tag{3.11}$$

### A. Finiteness of Noise Kernel

We now turn our attention to the noise kernel. First introduced in the context of stochastic semiclassical gravity it is the symmetrized product of the (mean subtracted) stress tensor operator:

$$\begin{aligned}
8N_{ab,c'd'}(x, y) = & \left\langle \left\{ \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle, \hat{T}_{c'd'}(y) - \langle \hat{T}_{c'd'}(y) \rangle \right\} \right\rangle \\
= & \left\langle \left\{ \hat{T}_{ab}(x), \hat{T}_{c'd'}(y) \right\} \right\rangle - 2 \langle \hat{T}_{ab}(x) \rangle \langle \hat{T}_{c'd'}(y) \rangle
\end{aligned} \tag{3.12}$$

Since  $\hat{T}_{ab}(x)$  defined at one point can be ill-behaved as it is generally divergent, one can question the soundness of these quantities. But as will be shown later, the noise kernel is finite for  $y \neq x$ . All field operator products present in the first expectation value that could be divergent are canceled by similar products in the second term. We will replace each of the stress tensor operators in the above expression for the noise kernel by their point separated versions,

effectively separating the two points  $(x, y)$  into the four points  $(x, x', y, y')$ . This will allow us to express the noise kernel in terms of a pair of differential operators acting on a combination of four and two point functions. Wick's theorem will allow the four point functions to be re-expressed in terms of two point functions. From this we see that all possible divergences for  $y \neq x$  will cancel. When the coincident limit is taken divergences do occur. The above procedure will allow us to isolate the divergences and obtain a finite result.

Taking the point-separated quantities as more basic, one should replace each of the stress tensor operators in the above with the corresponding point separated version (3.8), with  $\mathcal{T}_{ab}$  acting at  $x$  and  $x'$  and  $\mathcal{T}_{c'd'}$  acting at  $y$  and  $y'$ . In this framework the noise kernel is defined as

$$8N_{ab,c'd'}(x, y) = \lim_{x' \rightarrow x} \lim_{y' \rightarrow y} \mathcal{T}_{ab} \mathcal{T}_{c'd'} G(x, x', y, y') \quad (3.13)$$

where the four point function is

$$G(x, x', y, y') = \frac{1}{4} \left[ \langle \left\{ \hat{\phi}(x), \hat{\phi}(x') \right\}, \left\{ \hat{\phi}(y), \hat{\phi}(y') \right\} \rangle \right. \\ \left. - 2 \langle \left\{ \hat{\phi}(x), \hat{\phi}(x') \right\} \rangle \langle \left\{ \hat{\phi}(y), \hat{\phi}(y') \right\} \rangle \right]. \quad (3.14)$$

We assume the pairs  $(x, x')$  and  $(y, y')$  are each within their respective Riemann normal coordinate neighborhoods so as to avoid problems that possible geodesic caustics might be present. When we later turn our attention to computing the limit  $y \rightarrow x$ , after issues of regularization are addressed, we will want to assume all four points are within the same Riemann normal coordinate neighborhood.

Wick's theorem, for the case of free fields which we are considering, gives the simple product four point function in terms of a sum of products of Wightman functions (we use the shorthand notation  $G_{xy} \equiv G_+(x, y) = \langle \hat{\phi}(x) \hat{\phi}(y) \rangle$ ):

$$\langle \hat{\phi}(x) \hat{\phi}(y) \hat{\phi}(x') \hat{\phi}(y') \rangle = G_{xy'} G_{yx'} + G_{xx'} G_{yy'} + G_{xy} G_{x'y'} \quad (3.15)$$

Expanding out the anti-commutators in (3.14) and applying Wick's theorem, the four point function becomes

$$G(x, x', y, y') = G_{xy'} G_{x'y} + G_{xy} G_{x'y'} + G_{yx'} G_{y'x} + G_{yx} G_{y'x'} \quad (3.16)$$

We can now easily see that the noise kernel defined via this function is indeed well defined for the limit  $(x', y') \rightarrow (x, y)$ :

$$G(x, x, y, y) = 2 (G_{xy}^2 + G_{yx}^2), \quad (3.17)$$

From this we can see that the noise kernel is also well defined for  $y \neq x$ ; any divergence present in the first expectation value of (3.14) have been cancelled by those present in the pair of Green functions in the second term.

## B. Explicit Form of the Noise Kernel

We will let the points separated for a while so we can keep track of which covariant derivative acts on which arguments of which Wightman function. As an example (the complete calculation is quite long), consider the result of the first set of covariant derivative operators in the differential operator (3.5), from both  $\mathcal{T}_{ab}$  and  $\mathcal{T}_{c'd'}$ , acting on  $G(x, x', y, y')$ :

$$\frac{1}{4} (1 - 2\xi)^2 \left( g_a^{p''} \nabla_{p''} \nabla_b + g_b^{p''} \nabla_{p''} \nabla_a \right) \\ \times \left( g_c^{q'''} \nabla_{q'''} \nabla_{d'} + g_{d'}^{q'''} \nabla_{q'''} \nabla_{c'} \right) G(x, x', y, y') \quad (3.18)$$

(Our notation is that  $\nabla_a$  acts at  $x$ ,  $\nabla_{c'}$  at  $y$ ,  $\nabla_{b''}$  at  $x'$  and  $\nabla_{d''}$  at  $y'$ ). Expanding out the differential operator above, we can determine which derivatives act on which Wightman function:

$$\frac{(1 - 2\xi)^2}{4} \times \left[ g_c^{p'''} g^{q''}{}_a (G_{xy'; b p'''} G_{x'y; q'' d'} + G_{xy; b d'} G_{x'y'; q'' p'''} \right. \\ \left. + G_{yx'; q'' d'} G_{y'x; b p'''} + G_{yx; b d'} G_{y'x'; q'' p'''} \right) \\ + g_d^{p'''} g^{q''}{}_a (G_{xy'; b p'''} G_{x'y; q'' c'} + G_{xy; b c'} G_{x'y'; q'' p'''} \\ + G_{yx'; q'' c'} G_{y'x; b p'''} + G_{yx; b c'} G_{y'x'; q'' p'''} \\ \left. + g_c^{p'''} g^{q''}{}_b (G_{xy'; a p'''} G_{x'y; q'' d'} + G_{xy; a d'} G_{x'y'; q'' p'''} \right)$$

$$\begin{aligned}
& + G_{yx';q''d'} G_{y'x;ap'''} + G_{yx;ad'} G_{y'x';q''p'''} \\
& + g_{d'} p''' g^{q''}{}_b (G_{xy';ap'''} G_{x'y;q''c'} + G_{xy;ac'} G_{x'y';q''p'''} \\
& + G_{yx';q''c'} G_{y'x;ap'''} + G_{yx;ac'} G_{y'x';q''p'''})]
\end{aligned} \tag{3.19}$$

If we now let  $x' \rightarrow x$  and  $y' \rightarrow y$  the contribution to the noise kernel is (including the factor of  $\frac{1}{8}$  present in the definition of the noise kernel):

$$\begin{aligned}
& \frac{1}{8} \left\{ (1 - 2\xi)^2 (G_{xy;ad'} G_{xy;bc'} + G_{xy;ac'} G_{xy;bd'}) \right. \\
& \left. + (1 - 2\xi)^2 (G_{yx;ad'} G_{yx;bc'} + G_{yx;ac'} G_{yx;bd'}) \right\}
\end{aligned} \tag{3.20}$$

That this term can be written as the sum of a part involving  $G_{xy}$  and one involving  $G_{yx}$  is a general property of the entire noise kernel. It thus takes the form

$$N_{abc'd'}(x, y) = N_{abc'd'}[G_+(x, y)] + N_{abc'd'}[G_+(y, x)]. \tag{3.21}$$

We will present the form of the functional  $N_{abc'd'}[G]$  shortly. First we note, for  $x$  and  $y$  time-like separated, the above split of the noise kernel allows us to express it in terms of the Feynman (time ordered) Green function  $G_F(x, y)$  and the Dyson (anti-time ordered) Green function  $G_D(x, y)$ :

$$N_{abc'd'}(x, y) = N_{abc'd'}[G_F(x, y)] + N_{abc'd'}[G_D(x, y)] \tag{3.22}$$

<sup>7</sup> The complete form of the functional  $N_{abc'd'}[G]$  is

$$N_{abc'd'}[G] = \tilde{N}_{abc'd'}[G] + g_{ab}\tilde{N}_{c'd'}[G] + g_{c'd'}\tilde{N}'_{ab}[G] + g_{ab}g_{c'd'}\tilde{N}[G] \tag{3.24a}$$

with

$$\begin{aligned}
8\tilde{N}_{abc'd'}[G] = & (1 - 2\xi)^2 (G_{c'b} G_{d'a} + G_{c'a} G_{d'b}) + 4\xi^2 (G_{c'd'} G_{ab} + G G_{abc'd'}) \\
& - 2\xi (1 - 2\xi) (G_{b} G_{c'ad'} + G_{a} G_{c'bd'} + G_{d'} G_{abc'} + G_{c'} G_{abd'}) \\
& + 2\xi (1 - 2\xi) (G_{a} G_{b} R_{c'd'} + G_{c'} G_{d'} R_{ab}) \\
& - 4\xi^2 (G_{ab} R_{c'd'} + G_{c'd'} R_{ab}) G + 2\xi^2 R_{c'd'} R_{ab} G^2
\end{aligned} \tag{3.24b}$$

$$\begin{aligned}
8\tilde{N}'_{ab}[G] = & 2(1 - 2\xi) \left( \left( 2\xi - \frac{1}{2} \right) G_{p'b} G_{p'a} + \xi (G_{b} G_{p'a} + G_{a} G_{p'b}) \right) \\
& - 4\xi \left( \left( 2\xi - \frac{1}{2} \right) G_{p'} G_{abp'} + \xi (G_{p'} G_{ab} + G G_{abp'}) \right) \\
& - (m^2 + \xi R') ((1 - 2\xi) G_{a} G_{b} - 2G\xi G_{ab}) \\
& + 2\xi \left( \left( 2\xi - \frac{1}{2} \right) G_{p'} G_{p'} + 2G\xi G_{p'} \right) R_{ab} \\
& - (m^2 + \xi R') \xi R_{ab} G^2
\end{aligned} \tag{3.24c}$$

$$\begin{aligned}
8\tilde{N}[G] = & 2 \left( 2\xi - \frac{1}{2} \right)^2 G_{p'q} G_{p'q} + 4\xi^2 (G_{p'} G_{q} + G G_{p'q}) \\
& + 4\xi \left( 2\xi - \frac{1}{2} \right) (G_{p'} G_{q'} + G_{q'} G_{p'})
\end{aligned}$$

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<sup>7</sup>This can be connected with the zeta function approach to this problem [9] as follows: Recall when the quantum stress tensor fluctuations determined in the Euclidean section is analytically continued back to Lorentzian signature ( $\tau \rightarrow it$ ), the time ordered product results. On the other hand, if the continuation is  $\tau \rightarrow -it$ , the anti-time ordered product results. With this in mind, the noise kernel is seen to be related to the quantum stress tensor fluctuations derived via the effective action as

$$16N_{abc'd'} = \Delta T_{abc'd'}^2 \Big|_{t=-i\tau, t'=-i\tau'} + \Delta T_{abc'd'}^2 \Big|_{t=i\tau, t'=i\tau'}. \tag{3.23}$$

$$\begin{aligned}
& - \left( 2\xi - \frac{1}{2} \right) \left( (m^2 + \xi R) G_{;p'} G_{;p'}^{p'} + (m^2 + \xi R') G_{;p} G_{;p}^{p'} \right) \\
& - 2\xi \left( (m^2 + \xi R) G_{;p'}^{p'} + (m^2 + \xi R') G_{;p}^{p'} \right) G \\
& \frac{1}{2} (m^2 + \xi R) (m^2 + \xi R') G^2
\end{aligned} \tag{3.24d}$$

For a massless, conformally coupled scalar field ( $\xi = \frac{1}{6}$  and  $m = 0$ ), the noise kernel functional is

$$\begin{aligned}
72\tilde{N}_{abc'd'}[G] = & 4 (G_{;c'b} G_{;d'a} + G_{;c'a} G_{;d'b}) + G_{;c'd'} G_{;ab} + G G_{;abc'd'} \\
& - 2 (G_{;b} G_{;c'a d'} + G_{;a} G_{;c'b d'} + G_{;d'} G_{;abc'} + G_{;c'} G_{;abd'}) \\
& + 2 (G_{;a} G_{;b} R_{c'd'} + G_{;c'} G_{;d'} R_{ab}) \\
& - (G_{;ab} R_{c'd'} + G_{;c'd'} R_{ab}) G - \frac{1}{2} R_{c'd'} R_{ab} G^2
\end{aligned} \tag{3.25a}$$

$$\begin{aligned}
288\tilde{N}'_{ab}[G] = & 8 \left( -G_{;p'b} G_{;p'a}^{p'} + G_{;b} G_{;p'a}^{p'} + G_{;a} G_{;p'b}^{p'} \right) \\
& 4 \left( G_{;p'} G_{;abp'} - G_{;p'}^{p'} G_{;ab} + G G_{;abp'}^{p'} \right) \\
& - 2 R' (2 G_{;a} G_{;b} - G G_{;ab}) \\
& - 2 \left( G_{;p'} G_{;p'}^{p'} - 2 G G_{;p'}^{p'} \right) R_{ab} - R' R_{ab} G^2
\end{aligned} \tag{3.25b}$$

$$\begin{aligned}
288\tilde{N}[G] = & 2 G_{;p'q} G_{;p'q}^{p'q} + 4 \left( G_{;p'}^{p'} G_{;q}^{q'} + G G_{;p'q}^{p'q'} \right) \\
& - 4 \left( G_{;p} G_{;q'}^{p'q'} + G_{;q}^{p'} G_{;p'}^{q'} \right) \\
& + R G_{;p'} G_{;p'}^{p'} + R' G_{;p} G_{;p}^{p'} \\
& - 2 \left( R G_{;p'}^{p'} + R' G_{;p}^{p'} \right) G + \frac{1}{2} R R' G^2
\end{aligned} \tag{3.25c}$$

For the minimal coupling ( $\xi = 0$ ) case:

$$8\tilde{N}_{abc'd'}[G] = G_{;c'b} G_{;d'a} + G_{;c'a} G_{;d'b} \tag{3.26a}$$

$$8\tilde{N}'_{ab}[G] = -G_{;p'b} G_{;p'a}^{p'} - m^2 G_{;a} G_{;b} \tag{3.26b}$$

$$8\tilde{N}[G] = \frac{1}{2} \left( G_{;p'q} G_{;p'q}^{p'q} + m^2 \left( G_{;p'} G_{;p'}^{p'} + G_{;p} G_{;p}^{p'} \right) + m^4 G^2 \right) \tag{3.26c}$$

### C. Trace of the Noise Kernel

One of the most interesting and surprising results to come out of the investigations undertaken in the 1970's of the quantum stress tensor was the discovery of the trace anomaly [36]. When the trace of the stress tensor  $T = g^{ab} T_{ab}$  is evaluated for a field configuration that satisfies the field equation

$$\frac{\delta S[\phi]}{\delta \phi(x)} = 0 \Rightarrow (\square - \xi R - m^2) \phi = 0, \tag{3.27}$$

the trace is seen to vanish for the massless conformally coupled case. When this analysis is carried over to the renormalized expectation value of the quantum stress tensor, the trace no longer vanishes. Wald [28] showed this was due to the failure of the renormalized Hadamard function  $G_{\text{ren}}(x, x')$  to be symmetric in  $x$  and  $x'$ , implying it does not necessarily satisfy the field equation (3.27) in the variable  $x'$ . (We discuss in the next section the definition of  $G_{\text{ren}}(x, x')$  in the context of point separation.)

With this in mind, we can now determine the noise associated with the trace. Taking the trace at both points  $x$  and  $y$  of the noise kernel functional (3.22):

$$\begin{aligned}
N[G] &= g^{ab} g^{c'd'} N_{abc'd'}[G] \\
&= -3G\xi \left\{ \left( m^2 + \frac{1}{2}\xi R \right) G_{;p'}{}^{p'} + \left( m^2 + \frac{1}{2}\xi R' \right) G_{;p}{}^p \right\} \\
&\quad + \frac{9\xi^2}{2} \left\{ G_{;p'}{}^{p'} G_{;p}{}^p + G G_{;p}{}^p G_{;p'}{}^{p'} \right\} + \left( m^2 + \frac{1}{2}\xi R \right) \left( m^2 + \frac{1}{2}\xi R' \right) G^2 \\
&\quad + 3 \left( \frac{1}{6} - \xi \right) \left\{ +3 \left( \frac{1}{6} - \xi \right) G_{;p'} G_{;p}{}^{p'} - 3\xi \left( G_{;p} G_{;p'}{}^{pp'} + G_{;p'} G_{;p}{}^{pp'} \right) \right. \\
&\quad \left. \left( m^2 + \frac{1}{2}\xi R \right) G_{;p'} G_{;p}{}^{p'} + \left( m^2 + \frac{1}{2}\xi R' \right) G_{;p} G_{;p}{}^p \right\}
\end{aligned} \tag{3.28}$$

For the massless conformal case, this reduces to

$$N[G] = \frac{1}{144} \{ RR'G^2 - 6G(R\Box' + R'\Box)G + 18((\Box G)(\Box' G) + \Box'\Box G) \} \tag{3.29}$$

which holds for any function  $G(x, y)$ . For  $G$  being the Green function, it satisfies the field equation (3.27):

$$\Box G = (m^2 + \xi R)G \tag{3.30}$$

We will only assume the Green function satisfies the field equation in its first variable. Using the fact  $\Box'R = 0$  (because the covariant derivatives act at a different point than at which  $R$  is supported), it follows that

$$\Box'\Box G = (m^2 + \xi R)\Box'G. \tag{3.31}$$

With these results, the noise kernel trace becomes

$$\begin{aligned}
N[G] &= \frac{1}{2} \left( m^2 (1 - 3\xi) + 3R \left( \frac{1}{6} - \xi \right) \xi \right) \\
&\quad \times \left\{ G^2 (2m^2 + R'\xi) + (1 - 6\xi) G_{;p'} G_{;p}{}^{p'} - 6G\xi G_{;p'}{}^{p'} \right\} \\
&\quad + \frac{1}{2} \left( \frac{1}{6} - \xi \right) \left\{ 3 (2m^2 + R'\xi) G_{;p} G_{;p}{}^p - 18\xi G_{;p} G_{;p'}{}^{pp'} \right. \\
&\quad \left. + 18 \left( \frac{1}{6} - \xi \right) G_{;p'} G_{;p}{}^{p'} \right\}
\end{aligned} \tag{3.32}$$

It vanishes for the massless conformal case. We have thus shown, based solely on the definition of the point separated noise kernel, there is no noise associated with the trace anomaly. Our result is completely general since we have assumed the Green function is only satisfying the field equations in its first variable. This condition holds not just for the classical field case, but also for the regularized quantum case, where we do not expect the Green function to satisfy the field equation in both variables. One can see this result from a simple observation: Since the trace anomaly is known to be locally determined and quantum state independent, whereas the noise present in the quantum field is non-local, it is hard to find a noise associated with it. This is in agreement with previous findings [11,13,14], derived from the Feynman-Vernon influence functional formalism [15].

#### IV. REGULARIZATION OF THE NOISE KERNEL

We pointed out earlier that field quantities defined at two separated points possess important information which could be the starting point for probes into possible extended structures of spacetime. Moving in the other (homeward) direction, it is of interest to see how fluctuations of energy momentum (loosely, noise) would enter in the ordinary (point-wise) quantum field theory in helping us to address a new set of issues such as a) whether the fluctuations to mean ratio can act as a criterion for the validity of semiclassical (note it is not that simple, see [10]), b) whether the fluctuations in the vacuum energy density which drives inflationary violates the positive energy condition, c) deriving structure formation from quantum fluctuations, or d) deriving general relativity as a low energy effective theory in the geometro-hydrodynamic limit [6]. For these inquires we need to examine the consequences of taking or reaching the coincident limit and to construct regularization procedures to treat the ultraviolet divergences in order to obtain a finite result for the noise kernel in this limit.

We can see from the point separated form of the stress tensor (3.10) what we need to regularize is the Green function  $G^{(1)}(x, x')$ . Once the Green function has been regularized such that it is smooth and has a well defined  $x' \rightarrow x$  limit,

the stress tensor will be well defined. In Minkowski space, this issue is easily resolved by a "normal ordering" prescription, which hinges on the existence of a unique vacuum. Unfortunately, for a general curved spacetime, there is no unique vacuum, and hence, no unique mode expansion on which to build a normal ordering prescription. But we can still ask if there is a way to determine a contribution we can subtract to yield a unique quantum stress tensor. Here we follow the prescription of Wald [26], and Adler *et. al.* [27] (with corrections [28]) summarized in [3]. We give a short synopsis below as it will be referred to in subsequent discussions and later papers.

The idea builds on the fact that for  $G(x, x')_\omega = \langle \omega | \hat{\phi}(x) \hat{\phi}(x') | \omega \rangle$ , the function

$$F(x, x') = G(x, x')_{\omega_1} - G(x, x')_{\omega_2} \quad (4.1)$$

is a smooth function of  $x$  and  $x'$ , where  $\omega_1$  and  $\omega_2$  denote two different states. This means the difference between the stress tensor for two states is well defined for the point separation scheme, *i.e.*,

$$\mathcal{F}_{ab} = \frac{1}{2} \lim_{x' \rightarrow x} \mathcal{T}_{ab} (F(x, x') + F(x', x)) \quad (4.2)$$

is well defined. So a bi-distribution  $G^L(x, x')$  might be useful for the vacuum subtraction. At first, it would seem unlikely we could find such a unique bi-distribution. Wald found that with the introduction of four axioms for the regularized stress tensor

$$\langle \hat{T}_{ab}(x) \rangle_{\text{ren}} = \lim_{x' \rightarrow x} \frac{1}{2} \mathcal{T}_{ab} \left( G^{(1)}(x, x') - G^L(x, x') \right) \quad (4.3)$$

$G^L(x, x')$  is uniquely determined, up to a local conserved curvature term. The Wald axioms are [28,3]:

1. The difference between the stress tensor for two states should agree with (4.2);
2. The stress tensor should be local with respect to the state of the field;
3. For all states, the stress tensor is conserved:  $\nabla^a \langle T_{ab} \rangle = 0$ ;
4. In Minkowski space, the result  $\langle 0 | T_{ab} | 0 \rangle = 0$  is recovered.

We are still left with the problem of determining the form of such a bi-distribution. Hadamard [37] showed that the Green function for a large class of states takes the form (in four spacetime dimensions)

$$G^L(x, x') = \frac{1}{8\pi^2} \left( \frac{2U(x, x')}{\sigma} + V(x, x') \log \sigma + W(x, x') \right) \quad (4.4)$$

with  $U(x, x')$ ,  $V(x, x')$  and  $W(x, x')$  being smooth functions.<sup>8</sup> We refer to Eqn (4.4) as the "Hadamard ansatz".

Since the functions  $V(x, x')$  and  $W(x, x')$  are smooth functions, they can be expanded as

$$V(x, x') = \sum_{n=0}^{\infty} v_n(x, x') \sigma^n \quad (4.5a)$$

$$W(x, x') = \sum_{n=0}^{\infty} w_n(x, x') \sigma^n \quad (4.5b)$$

with the  $v_n$ 's and  $w_n$ 's themselves smooth functions. These functions and  $U(x, x')$  are determined by substituting  $G^L(x, x')$  in the wave equation  $KG^L(x, x') = 0$  and equating to zero the coefficients of the explicitly appearing powers of  $\sigma^n$  and  $\sigma^n \log \sigma$ . Doing so, we get the infinite set of equations

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<sup>8</sup>When working in the Lorentz sector of a field theory, *i.e.*, when the metric signature is  $(-, +, +, +)$ , as opposed to the Euclidean sector with the signature  $(+, +, +, +)$ , we must modify the above function to account for null geodesics, since  $\sigma(x, x') = 0$  for null separated  $x$  and  $x'$ . This problem can be overcome by using  $\sigma \rightarrow \sigma + 2i\epsilon(t - t') + \epsilon^2$ . Here, we will work only with geometries that possess Euclidean sectors and carry out our analysis with Riemannian geometries and only at the end continue back to the Lorentzian geometry. Nonetheless, this presents no difficulty. At any point in the analysis the above replacement for  $\sigma$  can be performed.

$$U(x, x') = \Delta^{1/2}; \quad (4.6a)$$

$$2H_0 v_0 + K \Delta^{1/2} = 0; \quad (4.6b)$$

$$2nH_n v_n + K v_{n-1} = 0, \quad n \geq 1; \quad (4.6c)$$

$$2H_{2n} v_n + 2nH_n w_n + K w_{n-1} = 0, \quad n \geq 1 \quad (4.6d)$$

with

$$H_n = \sigma^{ip} \nabla_p + \left( n - 1 + \frac{1}{2} (\square \sigma) \right) \quad (4.6e)$$

From Eqs (4.6d), the functions  $v_n$  are completely determined. In fact, they are symmetric functions, and hence  $V(x, x')$  is a symmetric function of  $x$  and  $x'$ . On the other hand, the field equations only determine  $w_n$ ,  $n \geq 1$ , leaving  $w_0(x, x')$  completely arbitrary. This reflects the state dependence of the Hadamard form above. Moreover, even if  $w_0(x, x')$  is chosen to be symmetric, this does not guarantee that  $W(x, x')$  will be. By using axiom (4)  $w_0(x, x') \equiv 0$ . With this choice, the Minkowski spacetime limit is

$$G^L = \frac{1}{(2\pi)^2} \frac{1}{\sigma} \quad (4.7)$$

where now  $2\sigma = (t - t')^2 - (\mathbf{x} - \mathbf{x}')^2$  and this corresponds to the correct vacuum contribution that needs to be subtracted.

With this choice though, we are left with a  $G^L(x, x')$  which is not symmetric and hence does not satisfy the field equation at  $x'$ , for fixed  $x$ . Wald [28] showed this in turn implies axiom (3) is not satisfied. He resolved this problem by adding to the regularized stress tensor a term which cancels that which breaks the conservation of the old stress tensor:

$$\langle T_{ab}^{\text{new}} \rangle = \langle T_{ab}^{\text{old}} \rangle + \frac{1}{2(4\pi)^2} g_{ab} [v_1] \quad (4.8)$$

where  $[v_1] = v_1(x, x)$  is the coincident limit of the  $n = 1$  solution of Eq (4.6d). This yields a stress tensor which satisfies all four axioms and produces the well known trace anomaly  $\langle T_a^a \rangle = [v_1] / 8\pi^2$ . We can view this redefinition as taking place at the level of the stress tensor operator via

$$\hat{T}_{ab} \rightarrow \hat{T}_{ab} + \frac{\hat{1}}{2(4\pi)^2} g_{ab} [v_1] \quad (4.9)$$

Since this amounts to a constant shift of the stress tensor operator, it will have no effect on the noise kernel or fluctuations, as they are the variance about the mean. This is further supported by the fact that there is no noise associated with the trace. Since this result was derived by only assuming that the Green function satisfies the field equation in *one* of its variables, it is independent of the issue of the lack of symmetry in the Hadamard ansatz (4.4).

Using the above formalism we now derive the coincident limit expression for the noise kernel (3.24). To get a meaningful result, we must work with the regularization of the Wightman function, obtained by following the same procedure outlined above for the Hadamard function:

$$G_{\text{ren}}(x, y) \equiv G_{\text{ren},+}(x, y) = G_+(x, y) - G^L(x, y) \quad (4.10)$$

In doing this, we assume the singular structure of the Wightman function is the same as that for the Hadamard function. In all applications, this is indeed the case. Moreover, when we compute the coincident limit of  $N_{abc'd'}$ , we will be working in the Euclidean section where there is no issue of operator ordering. For now we only consider spacetimes with no time dependence present in the final coincident limit result, so there is also no issue of Wick rotation back to a Minkowski signature. If this was the case, then care must be taken as to whether we are considering  $[N_{abc'd'} [G_{\text{ren},+}(x, y)]]$  or  $[N_{abc'd'} [G_{\text{ren},+}(y, x)]]$ .

We now have all the information we need to compute the coincident limit of the noise kernel (3.24). Since the point separated noise kernel  $N_{abc'd'}(x, y)$  involves covariant derivatives at the two points at which it has support, when we take the coincident limit we can use Synge's theorem (2.3) to move the derivatives acting at  $y$  to ones acting at  $x$ . Due to the long length of the noise kernel expression, we will only give an example by examining a single term.

Consider a typical term from the noise kernel functional (3.24):

$$G_{\text{ren};c'b} G_{\text{ren};d'a} + G_{\text{ren};c'a} G_{\text{ren};d'b} \quad (4.11)$$

Recall the noise kernel itself is related to the noise kernel functional via

$$N_{abc'd'} = N_{abc'd'} [G_{ren}(x, y)] + N_{abc'd'} [G_{ren}(y, x)]. \quad (4.12)$$

This is implemented on our typical term by adding to it the same term, but now with the roles of  $x$  and  $y$  reversed, so we have to consider

$$G_{ren;c'b} G_{ren;d'a} + G_{ren;a'd} G_{ren;b'c} + G_{ren;c'a} G_{ren;d'b} + G_{ren;a'c} G_{ren;b'd} \quad (4.13)$$

It is to this form that we can take the coincident limit:

$$[G_{ren;c'b}] [G_{ren;d'a}] + [G_{ren;a'd}] [G_{ren;b'c}] + [G_{ren;c'a}] [G_{ren;d'b}] + [G_{ren;a'c}] [G_{ren;b'd}] \quad (4.14)$$

We can now apply Sygne's theorem:

$$\begin{aligned} & ([G_{ren;a};d] - [G_{ren;a}d]) ([G_{ren;b};c] - [G_{ren;b}c]) \\ & + ([G_{ren;d};a] - [G_{ren;a}d]) ([G_{ren;c};b] - [G_{ren;b}c]) \\ & + ([G_{ren;a};c] - [G_{ren;a}c]) ([G_{ren;b};d] - [G_{ren;b}d]) \\ & + ([G_{ren;c};a] - [G_{ren;a}c]) ([G_{ren;d};b] - [G_{ren;b}d]). \end{aligned} \quad (4.15)$$

This is the desired form for once we have an end point expansion of  $G_{ren}$ , it will be straightforward to compute the above expression. We will discuss the details of such an evaluation in the context of symbolic computations in a companion paper in this series.

The final result for the coincident limit of the noise kernel is broken down into a rank four and rank two tensor and a scalar according to

$$[N_{abc'd'}] = N'_{abcd} + g_{ab} N''_{cd} + g_{cd} N''_{ab} + g_{ab} g_{cd} N''' \quad (4.16a)$$

The two tensors and the scalar have the following form in terms of the coincident limit and derivatives of the renormalized Green function and its derivatives:

$$\begin{aligned} 8N_{abcd} = & (1 - 2\xi)^2 \{ ([G_{ren;a};c] - [G_{ren;a}c]) ([G_{ren;b};d] - [G_{ren;b}d]) \\ & + ([G_{ren;c};a] - [G_{ren;a}c]) ([G_{ren;d};b] - [G_{ren;b}d]) \\ & + ([G_{ren;a};d] - [G_{ren;a}d]) ([G_{ren;b};c] - [G_{ren;b}c]) \\ & + ([G_{ren;d};a] - [G_{ren;a}d]) ([G_{ren;c};b] - [G_{ren;b}c]) \} \\ & + 4\xi^2 \{ (-[G_{ren;a};b] - [G_{ren;b};a] + [G_{ren};ab] + [G_{ren;ab}]) [G_{ren;cd}] \\ & + [G_{ren;ab}] (-[G_{ren;c};d] - [G_{ren;d};c] + [G_{ren};cd] + [G_{ren;cd}]) \\ & + [G_{ren}] (-[G_{ren;abc};d] - [G_{ren;abd};c] + [G_{ren;ab};cd] + [G_{ren;abdc}]) \\ & + [G_{ren}] (-[G_{ren;cda};b] - [G_{ren;cdb};a] + [G_{ren;cd};ab] + [G_{ren;cdba}]) \} \\ & - 2\xi (1 - 2\xi) \{ ([G_{ren};a] - [G_{ren;a}]) ([G_{ren;cd};b] - [G_{ren;cdb}]) \\ & + ([G_{ren};b] - [G_{ren;b}]) ([G_{ren;cd};a] - [G_{ren;cda}]) \\ & + ([G_{ren};c] - [G_{ren;c}]) ([G_{ren;ab};d] - [G_{ren;abd}]) \\ & + ([G_{ren};d] - [G_{ren;d}]) ([G_{ren;ab};c] - [G_{ren;abc}]) \\ & + [G_{ren;a}] (-[G_{ren;bc};d] - [G_{ren;bd};c] + [G_{ren;b};cd] + [G_{ren;bdc}]) \\ & + [G_{ren;b}] (-[G_{ren;ac};d] - [G_{ren;ad};c] + [G_{ren;a};cd] + [G_{ren;adc}]) \\ & + [G_{ren;c}] (-[G_{ren;ad};b] - [G_{ren;bd};a] + [G_{ren;d};ab] + [G_{ren;bda}]) \\ & + [G_{ren;d}] (-[G_{ren;ac};b] - [G_{ren;bc};a] + [G_{ren;c};ab] + [G_{ren;bca}]) \} \\ & + 2\xi (1 - 2\xi) \{ ([G_{ren};a] - [G_{ren;a}]) ([G_{ren};b] - [G_{ren;b}]) + [G_{ren;a}] [G_{ren;b}] R_{cd} \\ & + ([G_{ren};c] - [G_{ren;c}]) ([G_{ren};d] - [G_{ren;d}]) + [G_{ren;c}] [G_{ren;d}] R_{ab} \} \\ & + 4\xi^2 \{ ([G_{ren;a};b] + [G_{ren;b};a] - [G_{ren};ab] - 2[G_{ren;ab}]) R_{cd} \\ & + ([G_{ren;c};d] + [G_{ren;d};c] - [G_{ren};cd] - 2[G_{ren;cd}]) R_{ab} \\ & + R_{ab} R_{cd} [G_{ren}] \} [G_{ren}] \end{aligned} \quad (4.16b)$$

$$\begin{aligned} 8N'_{ab} = & 2 \left( 2\xi - \frac{1}{2} \right) (1 - 2\xi) \{ ([G_{ren;b};p] - [G_{ren;pb}]) ([G_{ren;a}]^p - [G_{ren;a}^p]) \\ & + ([G_{ren;p};b] - [G_{ren;pb}]) ([G_{ren}]^p;_a - [G_{ren;a}^p]) \} \\ & + 2\xi (1 - 2\xi) \{ ([G_{ren};a] - [G_{ren;a}]) ([G_{ren;p}^p;_b] - [G_{ren;p}^p_b]) \\ & + ([G_{ren};b] - [G_{ren;b}]) ([G_{ren;p}^p;_a] - [G_{ren;p}^p_a]) \\ & + [G_{ren;a}] (-[G_{ren;pb}]^p - [G_{ren;b}^p]_p + [G_{ren;b};p]^p + [G_{ren;b}^p_p]) \} \end{aligned}$$



$$\begin{aligned}
& + [G_{ren}; b] (-[G_{ren}; pa]^{;p} - [G_{ren}; a^p]_{;p} + [G_{ren}; a]_{;p}^p + [G_{ren}; a^p_p]) \\
& - 4\xi \left( 2\xi - \frac{1}{2} \right) ([G_{ren};^ip] (-[G_{ren}; pa]_{;b} - [G_{ren}; pb]_{;a} + [G_{ren}; p]_{;ab} + [G_{ren}; pba]) \\
& \quad + ([G_{ren};^ip] - [G_{ren};^ip]) ([G_{ren}; ab]_{;p} - [G_{ren}; abp])) \\
& - 2(m^2 + \xi R) \xi [G_{ren}] ([G_{ren}; a]_{;b} + [G_{ren}; b]_{;a} - [G_{ren}; ab] - 2[G_{ren}; ab]) \\
& + 2\xi \left( 2\xi - \frac{1}{2} \right) ([G_{ren};_p] [G_{ren};^ip] - 2[G_{ren};_p] [G_{ren};^ip] + 2[G_{ren};_p] [G_{ren};^ip]) R_{ab} \\
& - 4\xi^2 [G_{ren}] (2[G_{ren};_p]^{;p} - [G_{ren};_p]^p - 2[G_{ren};_p]^p) R_{ab} \\
& - 2(m^2 + \xi R) \xi R_{ab} [G_{ren}]^2
\end{aligned} \tag{4.16c}$$

$$\begin{aligned}
8N''' &= 2 \left( 2\xi - \frac{1}{2} \right)^2 (([G_{ren};_p]_{;q} - [G_{ren}; pq]) ([G_{ren};^ip]_{;q} - [G_{ren};^ipq]) \\
& \quad + ([G_{ren};_q]_{;p} - [G_{ren}; pq]) ([G_{ren};^iq]_{;p} - [G_{ren};^ipq])) \\
& - 8\xi^2 ((2[G_{ren};_p]^{;p} - [G_{ren};_p]^p - [G_{ren};_p]^p) [G_{ren};^q] \\
& \quad [G_{ren}] (2[G_{ren};_p^p]_{;q} - [G_{ren};_p^p]_{;q} - [G_{ren};_p^p]_{;q})) \\
& - 8\xi \left( 2\xi - \frac{1}{2} \right) (([G_{ren};_p] - [G_{ren};_p]) [G_{ren}; q^{qp}] - [G_{ren};_p] [G_{ren}; q^{q}]^{;p} \\
& \quad + [G_{ren};_p] (2[G_{ren}; q^{qp}]_{;q} + [G_{ren}; q^{q}]_{;p} - [G_{ren};^ip]_{;q} - [G_{ren}; q^{pq}])) \\
& - 2(m^2 + \xi R) \left( 2\xi - \frac{1}{2} \right) ([G_{ren};_p] [G_{ren};^ip] - 2[G_{ren};_p] [G_{ren};^ip] + 2[G_{ren};_p] [G_{ren};^ip]) \\
& + 4(m^2 + \xi R) \xi (2[G_{ren};_p]^{;p} - [G_{ren};_p]^p - 2[G_{ren};_p]^p) [G_{ren}] \\
& + (m^2 + \xi R)^2 [G_{ren}]^2
\end{aligned} \tag{4.16d}$$

For the conformal coupling ( $\xi = \frac{1}{6}$  and  $m = 0$ ) case:

$$\begin{aligned}
72N_{abcd} &= 4 \{ ([G_{ren}; a]_{;c} - [G_{ren}; ac]) ([G_{ren}; b]_{;d} - [G_{ren}; bd]) \\
& \quad + ([G_{ren}; c]_{;a} - [G_{ren}; ac]) ([G_{ren}; d]_{;b} - [G_{ren}; bd]) \\
& \quad + ([G_{ren}; a]_{;d} - [G_{ren}; ad]) ([G_{ren}; b]_{;c} - [G_{ren}; bc]) \\
& \quad + ([G_{ren}; d]_{;a} - [G_{ren}; ad]) ([G_{ren}; c]_{;b} - [G_{ren}; bc]) \} \\
& + \{ (-[G_{ren}; a]_{;b} - [G_{ren}; b]_{;a} + [G_{ren}; ab] + [G_{ren}; ab]) [G_{ren}; cd] \\
& \quad + [G_{ren}; ab] (-[G_{ren}; c]_{;d} - [G_{ren}; d]_{;c} + [G_{ren}; cd] + [G_{ren}; cd]) \\
& \quad + [G_{ren}] (-[G_{ren}; abc]_{;d} - [G_{ren}; abd]_{;c} + [G_{ren}; ab]_{;cd} + [G_{ren}; abcd]) \\
& \quad + [G_{ren}] (-[G_{ren}; cda]_{;b} - [G_{ren}; cdb]_{;a} + [G_{ren}; cd]_{;ab} + [G_{ren}; cdba]) \} \\
& - 2 \{ ([G_{ren}; a] - [G_{ren}; a]) ([G_{ren}; cd]_{;b} - [G_{ren}; cdb]) \\
& \quad + ([G_{ren}; b] - [G_{ren}; b]) ([G_{ren}; cd]_{;a} - [G_{ren}; cda]) \\
& \quad + ([G_{ren}; c] - [G_{ren}; c]) ([G_{ren}; ab]_{;d} - [G_{ren}; abd]) \\
& \quad + ([G_{ren}; d] - [G_{ren}; d]) ([G_{ren}; ab]_{;c} - [G_{ren}; abc]) \\
& \quad + [G_{ren}; a] (-[G_{ren}; bc]_{;d} - [G_{ren}; bd]_{;c} + [G_{ren}; b]_{;cd} + [G_{ren}; bdc]) \\
& \quad + [G_{ren}; b] (-[G_{ren}; ac]_{;d} - [G_{ren}; ad]_{;c} + [G_{ren}; a]_{;cd} + [G_{ren}; adc]) \\
& \quad + [G_{ren}; c] (-[G_{ren}; ad]_{;b} - [G_{ren}; bd]_{;a} + [G_{ren}; d]_{;ab} + [G_{ren}; bda]) \\
& \quad + [G_{ren}; d] (-[G_{ren}; ac]_{;b} - [G_{ren}; bc]_{;a} + [G_{ren}; c]_{;ab} + [G_{ren}; bca]) \} \\
& + 2 \{ (([G_{ren}; a] - [G_{ren}; a]) ([G_{ren}; b] - [G_{ren}; b]) + [G_{ren}; a] [G_{ren}; b]) R_{cd} \\
& \quad + (([G_{ren}; c] - [G_{ren}; c]) ([G_{ren}; d] - [G_{ren}; d]) + [G_{ren}; c] [G_{ren}; d]) R_{ab} \} \\
& + \{ ([G_{ren}; a]_{;b} + [G_{ren}; b]_{;a} - [G_{ren}; ab] - 2[G_{ren}; ab]) R_{cd} \\
& \quad + ([G_{ren}; c]_{;d} + [G_{ren}; d]_{;c} - [G_{ren}; cd] - 2[G_{ren}; cd]) R_{ab} \\
& \quad + R_{ab} R_{cd} [G_{ren}] \} [G_{ren}]
\end{aligned} \tag{4.17a}$$

$$\begin{aligned}
144N'_{ab} &= -4 (([G_{ren}; b]_{;p} - [G_{ren}; pb]) ([G_{ren}; a]^{;p} - [G_{ren}; a^p]) \\
& \quad + ([G_{ren}; p]_{;b} - [G_{ren}; pb]) ([G_{ren};^ip]_{;a} - [G_{ren}; a^p]_{;a})) \\
& + 4 (([G_{ren}; a] - [G_{ren}; a]) ([G_{ren};^ip]_{;b} - [G_{ren};^ipb]) \\
& \quad + ([G_{ren}; b] - [G_{ren}; b]) ([G_{ren};^ip]_{;a} - [G_{ren};^ip a]) \\
& \quad + [G_{ren}; a] (-[G_{ren}; pb]^{;p} - [G_{ren}; b^p]_{;p} + [G_{ren}; b]_{;p}^p + [G_{ren}; b^p_p]) \\
& \quad + [G_{ren}; b] (-[G_{ren}; pa]^{;p} - [G_{ren}; a^p]_{;p} + [G_{ren}; a]_{;p}^p + [G_{ren}; a^p_p])) \\
& 2 ([G_{ren};^ip] (-[G_{ren}; pa]_{;b} - [G_{ren}; pb]_{;a} + [G_{ren}; p]_{;ab} + [G_{ren}; pba])
\end{aligned}$$

$$\begin{aligned}
& + ([G_{ren}]^p - [G_{ren}]^p) ([G_{ren;ab}]_p - [G_{ren;ab}]_p) \\
& - (R[G_{ren}] ([G_{ren;a}]_b + [G_{ren;b}]_a - [G_{ren;ab}] - 2[G_{ren;ab}])) \\
& - (([G_{ren}]_p [G_{ren}]^p - 2[G_{ren}]_p [G_{ren}]^p + 2[G_{ren;p}] [G_{ren}]^p) R_{ab}) \\
& - 2[G_{ren}] (2[G_{ren;p}]^p - [G_{ren}]_p^p - 2[G_{ren;p}]^p) R_{ab} \\
& - (R R_{ab}) [G_{ren}]^2
\end{aligned} \tag{4.17b}$$

$$\begin{aligned}
288N''' = & 2((([G_{ren;p}]_q - [G_{ren;pq}]) ([G_{ren}]^p;^q - [G_{ren}]^{pq})) \\
& + ([G_{ren;q}]_p - [G_{ren;pq}]) ([G_{ren}]^q;^p - [G_{ren}]^{pq})) \\
& - 8((2[G_{ren;p}]^p - [G_{ren}]_p^p - [G_{ren;p}]^p) [G_{ren;q}]^q \\
& [G_{ren}] (2[G_{ren;p}^p q]^{q;^q} - [G_{ren;p}]_q^{q;^q} - [G_{ren;p}^p q]^{pq})) \\
& + 8((([G_{ren}]_p - [G_{ren;p}]) [G_{ren;q}]^{qp} - [G_{ren}]_p [G_{ren;q}]^{qp}) \\
& + [G_{ren;p}] (2[G_{ren;q}^p]^{q;^q} + [G_{ren;q}]^q;^p - [G_{ren;p}]_q^{q;^q} - [G_{ren;q}^p]^{pq})) \\
& 2R([G_{ren}]_p [G_{ren}]^p - 2[G_{ren}]_p [G_{ren}]^p + 2[G_{ren;p}] [G_{ren}]^p) \\
& + 4R(2[G_{ren;p}]^p - [G_{ren}]_p^p - 2[G_{ren;p}]^p) [G_{ren}] \\
& + R^2 [G_{ren}]^2
\end{aligned} \tag{4.17c}$$

For the minimal coupling ( $\xi = 0$ ) case:

$$\begin{aligned}
8N'_{abcd} = & ([G_{ren;a}]_c - [G_{ren;ac}]) ([G_{ren;b}]_d - [G_{ren;bd}]) \\
& + ([G_{ren;c}]_a - [G_{ren;ac}]) ([G_{ren;d}]_b - [G_{ren;bd}]) \\
& + ([G_{ren;a}]_d - [G_{ren;ad}]) ([G_{ren;b}]_c - [G_{ren;bc}]) \\
& + ([G_{ren;d}]_a - [G_{ren;ad}]) ([G_{ren;c}]_b - [G_{ren;bc}])
\end{aligned} \tag{4.18a}$$

$$\begin{aligned}
8N'_{ab} = & -([G_{ren;b}]_p - [G_{ren;pb}]) ([G_{ren;a}]^p - [G_{ren;a}^p]) \\
& - ([G_{ren;p}]_b - [G_{ren;pb}]) ([G_{ren;a}]^p;^a - [G_{ren;a}^p]) \\
& - m^2 ([G_{ren}]_a - [G_{ren;a}]) ([G_{ren}]_b - [G_{ren;b}]) \\
& - m^2 [G_{ren;a}] [G_{ren;b}]
\end{aligned} \tag{4.18b}$$

$$\begin{aligned}
8N''' = & \frac{1}{2}((([G_{ren;p}]_q - [G_{ren;pq}]) ([G_{ren}]^p;^q - [G_{ren}]^{pq})) \\
& + ([G_{ren;q}]_p - [G_{ren;pq}]) ([G_{ren}]^q;^p - [G_{ren}]^{pq})) \\
& + m^2 ([G_{ren}]_p [G_{ren}]^p - 2[G_{ren}]_p [G_{ren}]^p + 2[G_{ren;p}] [G_{ren}]^p) \\
& + m^4 [G_{ren}]^2
\end{aligned} \tag{4.18c}$$

## V. FURTHER DEVELOPMENTS

In this article we have derived a general expression for the noise kernel, or the vacuum expectation value of the stress energy bi-tensor for a quantum scalar field in a general curved space time using the point separation method. The general form is expressed as products of covariant derivatives of the quantum field's Green function. It is finite when the noise kernel is evaluated for distinct pairs of points (and non-null points for a massless field). We also have shown the trace of the noise kernel vanishes, confirming there is no noise associated with the trace anomaly. This holds regardless of issues of regularization of the noise kernel.

The noise kernel as a two point function of the stress energy tensor diverges as the pair of points are brought together, representing the “standard” ultraviolet divergence present in the quantum field theory. By using the modified point separation regularization method we render the field's Green function finite in the coincident limit. This in turn permits the derivation of the formal expression for the regularized coincident limit of the noise kernel.

When the Green function is available in closed analytic form one can carry out an end point expansion according to (2.5), displaying the ultraviolet divergence. Subtraction of the Hadamard ansatz (4.4), expressed as a series expansion, (4.5), will render this Green function finite in the coincident limit. With this, one can calculate the noise kernel for a variety of spacetimes, as will be developed in Papers II, III. Here we give an outline of the program in this series.

A common analytic approximation is the Gaussian [38], which, in ultrastatic spacetimes, provides a closed form expression for the Green function. In Paper II we make use of this form to evaluate the noise kernel for a massless scalar field in hot flat space and the Einstein universe. For hot flat space, the Gaussian Green function is exact, while for the Einstein universe, the exact Green function is known. We also work explicitly with the ultrastatic

metric conformal to the Schwarzschild metric. Though the approximate Green function is known to be a fairly good approximation for the stress tensor, *i.e.*, to second order, we find the approximation breaks down at fourth order (the noise kernel needs up to four covariant derivatives of the Green function). This manifest in the noise kernel trace computed under Gaussian approximation failing to vanish; indeed the magnitude of noise kernel becomes comparable to the components themselves. This form of the Green function fails to satisfy the field equations at fourth order.

In Paper III we take advantage of the simple conformal transformation property of the scalar field's Green function to compute the noise kernel for a thermal fields in a flat FRW Universe as it is obtainable from that of hot flat space by a time dependent scale factor, and likewise use the Einstein Universe result to obtain the noise kernel for a closed FRW Universe. The most interesting case is when we conformally transform to the Schwarzschild metric. Though our results based on the Gaussian approximation break down close to the event horizon, we can obtain reasonable results for the fluctuations of the stress tensor of the Hawking flux in the far field region and check with analytic results [39]. Along the way, we verify our procedure by explicitly re-deriving the Page result [35] for the stress tensor. We note that in Page's original work, the direct use of the conformal transformation was circumvented by "guessing" the solution to a functional differential equation. Our result is the first we know where the methodology of point separation was carried all the way through to the final result. That we get the known results is a check on our method and its correct implementation.

We presently outline how the work presented in this article are used to compute the coincident limit of the noise kernel. In future articles in this series we will present the details and results. We assume we have an analytic closed form expression for the Green function; we focus on the Gaussian approximation [38]. As mentioned above, this Green function is expanded in an end point series and the Hadamard ansatz is subtracted, generating the series expansion of the renormalized Green function. At this point we could directly substitute this expansion in our expression for the coincident limit of the noise kernel (4.16). The resulting expression becomes quite large and it is hard to glean any physical meaning from the resulting expressions. Instead we let this be the point in the problem when an explicit metric is chosen. Once this is done, in the context of the symbolic computation, it is straightforward to determine the component values of the Green function expansion tensors. For  $d = 4$ , there is at most 70 unique values. From this we can readily generate all the needed component values of the coincident limits of the covariant derivatives of the Green function, along with the covariant derivatives of the coincident limits. It is these explicitly evaluated tensors that are then substituted in (4.16) to get the final result.

We must stress that though all this is done on a computer, no numerical approximations are used; all work is done symbolically in terms of the explicit functional form of the metric and the parameters of the field. The final results are exact to the extent that the analytic form of the Green function is exact.

As will be seen in Paper II and III the basic procedures for generating the needed series expansions are recursive on the expansion order. For the noise kernel we need results up to fourth order in the separation distance. The well established work for the stress tensor is to second order. This provides a check of our code by verifying we always get the known results for the stress tensor expectation value. Once we know the second order recursion is correct, we know the algorithm is functioning as desired and the (new) fourth order terms are correct. This becomes particularly important when we consider metrics conformally related, as we get intermediate results of up to 1100 terms in length.

When we are interested in spacetimes for which we know the Green function in a conformally related space time, we have modified the above procedure. The simple conformal transformation property of Green functions allows us to get the Green function in the physical metric we are interested in from that in the conformally related metric (for the Gaussian approximation, this is the conformally optical metric). The main obstacle to overcome is the subtraction of the Hadamard ansatz. The divergent Green function is defined in terms of the optical metric while the Hadamard ansatz in terms of the physical metric. We need to re-express the transformed optical metric in terms of the physical metric. The defining equations for the geometric objects (*e.g.* Eqns 2.4 and 2.8) on the optical metric are transformed to the physical metric and recursively solved. Now the Green function series expansion can be written solely in terms of the physical metric. It is at this point that the symbolic computation environment reigns; the fourth order term in the expansion of the renormalized Green function has over 1100 terms. On the other hand, it is exactly in this context that we re-derive the Page [35] result for the stress tensor vacuum expectation value on the Schwarzschild black hole. As we stressed above, since all the series expansions used are recursively derived, so with the lower order results checked, we know the resulting expression is correct. Now having the general expansion of the renormalized Green function, we proceed as we outlined above and choose a metric and compute the coincident limit of the noise kernel.

Though our main focus in this article has been to lay out the groundwork for computing the coincident limit of the noise kernel, our result (3.24a) is completely general. This expression for the noise kernel only assumes a scalar field and can be used with or without first considering issues of renormalization of the Green function. Also, our result for the coincident limit, (4.16), holds regardless of the choice of Green function or metric. It only requires that the Green function had a meaningful coincident limit. It can become the starting point for numerical work on the noise kernel, an avenue worthy of some deliberation.

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